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# A Perturbation Method for Non-Linear Dispersive Waves with an Application to Water Waves

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SUMMARY
A multiple scale perturbation method is developed to obtain asymptotic evolution equations for slowly varying wave train solutions to non-linear dispersive wave problems. The method appears to give results which are a generalization of Whitham's theory on one hand and a generalization of the ray theory on the other hand. First an application is given to a non-linear Klein–Gordon equation, then the method is applied to two-dimensional water waves on water of finite depth (Stokes waves).

## 1. Introduction

In this paper a multiple scale perturbation method is developed to obtain asymptotic representations of typical solutions of non-linear dispersive wave problems, namely slowly varying wavetrains. For such a wavetrain the global quantities, such as the amplitude, wave number, frequency, mean wave height, etc. vary by a small fraction per wavelength or per period. Hence two basic scales may be indicated: a local scale in which the wave phenomenon is uniform (periodic) and a slow (stretched) scale in which the amplitude and other global properties of the wavetrain vary by relative order of magnitude one.

For linear dispersive wave problems the ray method (*cf.* Lewis [1]) has been developed to give asymptotic results for slowly varying wavetrains. For non-linear dispersive wave problems Whitham's averaged Lagrangian method ([2], [3], [4]) has found wide application. The justification of Whitham's method by formal perturbation methods by Luke [5] and Hoogstraten [6], [7] has shown that the results obtained by Whitham's method are equivalent to the lowest order results of an asymptotic multiple scale method based on the small modulation rate $1/K$ ($K \gg 1$) of the wavetrain as the fundamental small parameter. For wave problems involving a small parameter $\varepsilon$ characterizing the non-linear effect of finite amplitude, the Whitham theory corresponds to the investigation of wavetrains with modulation rate of order of magnitude $1/K$ very much smaller than $\varepsilon$.

Hence we may distinguish three cases:

(i) $\varepsilon K \gg 1$, corresponding to the Whitham theory with first an asymptotic expansion with respect to $1/K$ and then an asymptotic expansion with respect to $\varepsilon$ (*cf.* [6], [7]).

(ii) $\varepsilon K \ll 1$, corresponding to the ray theory with first an asymptotic expansion with respect to $\varepsilon$, yielding a linearized problem and then using an asymptotic expansion with respect to $1/K$, namely the well-known ray expansion ([1]).

(iii) $\varepsilon K = 1$, being the case where the small rate of modulation of the wavetrain is the same as the small parameter $\varepsilon$ characterizing the non-linearity of the problem.

In the sequel we will consider case (iii) which gives a generalization of the Whitham theory because the wavetrains to be studied may have a larger modulation rate than is allowed in the Whitham theory. This will be reflected by the appearance of some terms in the asymptotic evolution equations determining the slow variations of amplitude, wavenumber, etc., additional to the results from the Whitham theory. On the other hand the method of solution for case (iii) also appears to provide the generalization of the ray method to wave equations involving small non-linear terms.

In section 2 we will demonstrate our method by applying it to a relatively simple example, viz. a non-linear Klein–Gordon equation and we will derive asymptotically a set of three partial differential equations involving the amplitude, frequency and wavenumber of the wavetrain as functions of the slow variables. It appears that these equations have solutions representing wavetrains with permanent periodical envelopes, whereas the only class of wavetrains with permanent envelopes, being possible by the Whitham theory, is the class of uniform wavetrains.

In section 3 the method is applied to the problem of two-dimensional irrotational water waves on water of finite depth (Stokes waves). This problem which also has been treated by Chu and Mei [8], [9] in a somewhat different way, provides the appropriate generalization of the results of the Whitham theory derived in [3] and [7].

## 2. Asymptotic Solutions of a Non-Linear Klein-Gordon Equation

Consider the following non-linear Klein–Gordon equation for a one-dimensional wave function $\tilde{u}(\tilde{x}, \tilde{t})$:

$$\tilde{u}_{\tilde{t}\tilde{t}} - \tilde{u}_{\tilde{x}\tilde{x}} + V'(\tilde{u}) = 0\,, \quad V'(0) = 0\,. \tag{2.1}$$

For most physical applications $\tilde{u}(\tilde{x}, \tilde{t})$ is assumed to be small. Let us introduce the scaled wave function $u(\tilde{x}, \tilde{t})$ by means of

$$\tilde{u}(\tilde{x}, \tilde{t}) = \varepsilon u(\tilde{x}, \tilde{t})\,, \quad 0<\varepsilon \ll 1, \tag{2.2}$$

where $u = O(1)$ as $\varepsilon \to 0$ together with all its partial derivatives up to any order. Expanding the term $V'(\varepsilon u)$ in a Taylor series with respect to $\varepsilon u$ we find the equation

$$u_{\tilde{t}\tilde{t}} - u_{\tilde{x}\tilde{x}} + au + \varepsilon bu^2 + \varepsilon^2 cu^3 + \ldots = 0\,, \tag{2.3}$$

with $a$, $b$, $c$, etc. known constants depending on the form of $V'(.)$.

We want to consider slowly varying wavetrain solutions of eq. (2.3) with modulation rate $\varepsilon$. It is convenient to introduce the slow variables $x = \varepsilon\tilde{x}$ and $t = \varepsilon\tilde{t}$ to describe the slow variations of the wavetrain. Note that the wavelength and period are now $O(\varepsilon)$ in these new variables. Equation (2.3) becomes:

$$\varepsilon^2(u_{tt} - u_{xx}) + au + \varepsilon bu^2 + \varepsilon^2 cu^3 + \ldots = 0\,. \tag{2.4}$$

As a typical example we will consider the truncated equation

$$\varepsilon^2(u_{tt} - u_{xx}) + u + \varepsilon bu^2 = 0\,, \tag{2.5}$$

although the complete eq. (2.4) could be treated without difficulty by the method to be demonstrated. Furthermore $a$, $b$, $c$ etc. might depend on the slow variables $x$ and/or $t$ as well.

Considering the uniform wavetrain solution $u^*$ of equation (2.5):

$$u = u^*[\varepsilon^{-1}(\kappa x - \omega t)]\,, \tag{2.6}$$

where $u^*$ depends periodically on its argument and where $\kappa$ and $\omega$ denote the (constant) wavenumber and frequency, respectively, the wave fronts are given by the straight lines $\kappa x - \omega t = \text{constant}$. Looking for a slowly varying wavetrain $u(x, t; \varepsilon)$ the wavefronts can be characterized by curves $S(x, t; \varepsilon) = \text{constant}$, being those curves in the $(x, t)$-plane along which the normal derivative of $u$ is much larger than the tangential derivative. This condition is fulfilled by inserting for a slowly varying wavetrain solution of equation (2.5) a function of the form

$$u(x, t; \varepsilon) = U[\varepsilon^{-1}S(x, t; \varepsilon), x, t; \varepsilon]\,, \tag{2.7}$$

where $U$ depends *periodically* on the rapid variable

$$p = \varepsilon^{-1}S(x, t; \varepsilon)\,, \quad S \text{ phase function}\,.$$

The period of $U$ as a function of $p$ may be normalized to $2\pi$ and furthermore it is assumed that $U(p, x, t; \varepsilon)$ and $S(x, t; \varepsilon)$ are $O(1)$ as $\varepsilon\to 0$, together with all their partial derivatives with respect to $p$, $x$ and $t$ up to any order. The dependence of $U$ on $p$ gives the local periodic behaviour of the wavetrain, whereas the dependence of $U$ and $S$ on the slow variables $x$ and $t$ determines the modulation of amplitude, wave number, frequency, etc. The dependence of $S$ on $\varepsilon$ is inferred from the fact that the uniform wavetrain $u^*$ contains parameters $\kappa$ and $\omega$ depending on $\varepsilon$. This may be shown by evaluating $u^*$ exactly in inverse integral form, which is possible in the case of eq. (2.5).

Introducing the local wave number $\kappa(x, t; \varepsilon)$ and the local frequency $\omega(x, t; \varepsilon)$ by

$$\kappa = S_x\,, \quad \omega = -S_t\,, \quad \omega_x + \kappa_t = 0\,, \tag{2.8}$$

we substitute eq. (2.7) into eq. (2.5) to give

$$(\omega^2-\kappa^2)\,U_{pp} + U = 2\varepsilon(\omega U_{pt} + \kappa U_{px}) + \varepsilon(\omega_t + \kappa_x)\,U_p + \varepsilon^2(U_{xx} - U_{tt}) - \varepsilon b U^2\,. \tag{2.9}$$

It is essential in the multiple scale formalism to consider the variables $p$, $x$ and $t$ as *independent*. This means that equation (2.9) is a partial differential equation for $U$ as a function of $p$, $x$ and $t$, which, however, may be solved recursively as a set of ordinary differential equations for the dependence of $U$ on $p$ with $x$ and $t$ as parameters.

For the reduced case $\varepsilon=0$, equation (2.9) has the periodic solution

$$U = A\,\cos\frac{p}{(\omega^2-\kappa^2)^{\frac{1}{2}}}\,. \tag{2.10}$$

The normalization of the period to $2\pi$ then gives the dispersion relation

$$\omega^2-\kappa^2 = 1\,. \tag{2.11}$$

If equation (2.9) is considered as a non-linear ordinary differential equation for $U(p)$, we may use the basic idea of Lindstedt's (or Poincaré's) method (*cf.* Minorsky [10], Roseau [11]) in order to obtain the periodic solution. Analogous to Lindstedt's method we are motivated by the observation that a single expansion of $U$ in powers of $\varepsilon$:

$$U \sim U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots\,, \tag{2.12}$$

where the $U_i$ ($i=0, 1, 2, \dots$) are $O(1)$ together with all their partial derivatives with respect to $p$, $x$ and $t$ up to any order, leads to secular terms in the recurrent equations for the determination of $U_1$, $U_2$, etc. This is caused by the fact that the period of solutions of the non-reduced equation (2.9) is slightly different from the period of solutions of the reduced equation. We may expect the period of solutions of equation (2.9) to be of the form

$$2\pi\left[\frac{\omega^2-\kappa^2}{1+\varepsilon\Omega_1+\varepsilon^2\Omega_2+\dots}\right]^{\frac{1}{2}} \tag{2.13}$$

which after normalization to $2\pi$ yields the expansion

$$\omega^2-\kappa^2 \sim 1+\varepsilon\Omega_1+\varepsilon^2\Omega_2+\dots\,, \tag{2.14}$$

where the $\Omega_i$ ($i=1, 2, \dots$) are functions of $x$, $t$ and $\varepsilon$ being $O(1)$ as $\varepsilon\to 0$. Following Lindstedt's method we substitute *both* expansions (2.12) and (2.14) in the equation (2.9) for $U$. The additional freedom given by the unknown functions $\Omega_1$, $\Omega_2$, ..., can be used to suppress secular terms. In the case of a uniform wave we have $p=\varepsilon^{-1}\{\kappa(\varepsilon)x-\omega(\varepsilon)t\}$ and $U_t=U_x\equiv 0$. Then equation (2.9) reduces exactly to an ordinary differential equation for $U(p)$ and our asymptotic method is essentially the same as Lindstedt's method.

It should be stressed at this moment that the additional freedom provided by the introduction of the expansion (2.14) will appear to be not sufficient to suppress *all* secular terms which may arise in later stages of the computation. In this respect our problem is different from the ones involving those ordinary differential equations which may be solved approximately by Lindstedt's method.

Inserting the expansions (2.12) and (2.14) into the equation (2.9), we get

$$(1+\varepsilon\Omega_1+\varepsilon^2\Omega_2+\ldots)(U_{0pp}+\varepsilon U_{1pp}+\ldots)+U_0+\varepsilon U_1+\ldots=$$
$$=2\varepsilon\omega(U_{0pt}+\varepsilon U_{1pt}+\ldots)+2\varepsilon\kappa(U_{0px}+\varepsilon U_{1px}+\ldots)+$$
$$+\varepsilon(\kappa_x+\omega_t)(U_{0p}+\varepsilon U_{1p}+\ldots)+\varepsilon^2(U_{0xx}-U_{0tt}+\varepsilon U_{1xx}-\varepsilon U_{1tt}+\ldots)+$$
$$-\varepsilon b(U_0^2+2\varepsilon U_1U_0+\ldots).$$

Collecting terms of $O(1)$ and $O(\varepsilon)$ and equating them to zero, we obtain the equations:

$$U_{0pp}+U_0=0, \tag{2.15}$$

$$U_{1pp}+U_1=-\Omega_1U_{0pp}+2(\omega U_{0pt}+\kappa U_{0px})+(\omega_t+\kappa_x)U_{0p}-bU_0^2. \tag{2.16}$$

Without loss of generality we may take the solution for $U_0$ to be

$$U_0=A(x,t;\varepsilon)\cos p, \tag{2.17}$$

where the positive amplitude function $A$ is assumed to be $O(1)$ as $\varepsilon\to0$ together with its partial derivatives with respect to $x$ and $t$ up to any order. By allowing $A$ to depend on $\varepsilon$ and noting that $p$ may be shifted arbitrarily without changing the problem, we may require that no homogeneous solutions proportional to $\cos p$ and $\sin p$ can occur in the higher order perturbations $U_1$, $U_2$, etc. In this way we keep the expressions for $U_1$, $U_2$, etc. as simple as possible.

Using (2.17) the equation (2.16) for $U_1$ becomes:

$$U_{1pp}+U_1=A\Omega_1\cos p-2(\omega A_t+\kappa A_x)\sin p-(\omega_t+\kappa_x)A\sin p-\tfrac{1}{2}bA^2(1+\cos 2p). \tag{2.18}$$

In the righthand side of (2.18) the terms proportional to $\sin p$ and $\cos p$ are secular terms: they yield terms proportional to $p\cos p$ and $p\sin p$ in the solution for $U_1$. These terms are not periodic in $p$ and should be suppressed. Putting

$$\Omega_1=0, \tag{2.19}$$

the secular term with $\cos p$ in equation (2.18) vanishes. Furthermore we note that the equations for the higher order perturbations $U_i$ $(i=2, 3, \ldots)$ always contain a term of the form

$$A\Omega_i\cos p, \qquad (i=2,3,\ldots) \tag{2.20}$$

which is at our disposal to suppress secular terms with $\cos p$. However, if secular terms with $\sin p$ would arise, no means are available to suppress them. To overcome this difficulty new degrees of freedom should be introduced. This is accomplished by putting

$$2(\omega A_t+\kappa A_x)+(\omega_t+\kappa_x)A\sim\varepsilon\Lambda_1+\varepsilon^2\Lambda_2+\ldots, \tag{2.21}$$

where the $\Lambda_i$ $(i=1, 2, \ldots)$ are functions of $x$, $t$ and $\varepsilon$ being $O(1)$ as $\varepsilon\to0$. In this way we may omit the secular term with $\sin p$ from equation (2.18) for $U_1$ and in the subsequent equations for $U_i$ $(i=2, 3, \ldots)$ terms of the form $-\Lambda_{i-1}\sin p$ $(i=2, 3, \ldots)$ appear which may be used to suppress secular terms with $\sin p$ by making a suitable choice for $\Lambda_{i-1}$.

For $U_1$ we now obtain

$$U_1=-\tfrac{1}{2}bA^2+\tfrac{1}{6}bA^2\cos 2p. \tag{2.22}$$

The equation for $U_2$ is found to be

$$U_{2pp}+U_2=A\cos p\left[\Omega_2+\tfrac{5}{6}b^2A^2-\frac{1}{A}(A_{tt}-A_{xx})\right]-\Lambda_1\sin p+$$
$$-\tfrac{1}{6}b^2A^3\cos 3p-\tfrac{1}{3}bA[A(\kappa_x+\omega_t)+4(\omega A_t+\kappa A_x)]\sin 2p. \tag{2.23}$$

The secular terms in the righthand side of equation (2.23) should vanish. This gives

$$\Omega_2+\tfrac{5}{6}b^2A^2-\frac{1}{A}(A_{tt}-A_{xx})=0,\quad \Lambda_1=0, \tag{2.24}$$

and for $U_2$ we obtain $U_2 = \frac{1}{48}b^2 A^3 \cos 3p + \frac{1}{9}bA[A(\kappa_x + \omega_t) + 4(\omega A_t + \kappa A_x)] \sin 2p$.

Up to the present order of approximation we have obtained the following set of three partial differential equations for the slowly varying parameters $A$ (amplitude), $\kappa$ (wave number) and $\omega$ (frequency) as functions of the slow variables $x$ and $t$:

$$\omega^2 - \kappa^2 = 1 + \varepsilon^2 \left[ \frac{1}{A}(A_{tt} - A_{xx}) - \tfrac{5}{6}b^2 A^2 \right] + O(\varepsilon^3), \tag{2.25}$$

$$\frac{\partial}{\partial t}(\omega A^2) + \frac{\partial}{\partial x}(\kappa A^2) = O(\varepsilon^2), \tag{2.26}$$

$$\omega_x + \kappa_t = 0, \tag{2.27}$$

where the higher order terms $O(\varepsilon^2)$, $O(\varepsilon^3)$ etc. in equations (2.26) and (2.25) may be obtained by computing subsequently the secular terms in the equations for $U_3$, $U_4$, etc.

The equations (2.25), (2.26) and (2.27) form the generalization of the results by Whitham's theory to wavetrains with small modulation rate $\varepsilon$ per wavelength or period. Indeed, for wavetrains with modulation rate much smaller than the small parameter $\varepsilon$ characterizing the non-linearity of the problem, the terms involving derivatives of $A$ in equation (2.25) are negligible with respect to the other terms involving $A$, $\omega$ and $\kappa$ only. Then equation (2.25) reduces to Whitham's algebraical dispersion relation between $\omega$, $\kappa$ and $A$. The additional terms in (2.25) compared to Whitham's theory represent the influence of the slow modulation on the dispersion relation. They are not a result of the non-linearity of the problem. In fact, by putting $b=0$ we have a linear governing equation for which the asymptotic modulation equations (2.25), (2.26) and (2.27) still contain these additional terms. Remark that in the linear case ($b=0$) we have $U_0 = A \cos p$, $U_i \equiv 0$ ($i = 1, 2, 3, \ldots$) and the dispersion equation (2.25) then terminates after the $O(\varepsilon^2)$ term, and the righthand side of (2.26) becomes exactly zero.

One of the ways to solve equations (2.25), (2.26) and (2.27) approximately is to assume that the amplitude $A$ and the phase function $S$ are expandable in the form of an asymptotic power series in $\varepsilon$:

$$A(x, t; \varepsilon) \sim A_0(x, t) + \varepsilon A_1(x, t) + \varepsilon^2 A_2(x, t) + \ldots, \tag{2.28}$$

$$S(x, t; \varepsilon) \sim S_0(x, t) + \varepsilon S_1(x, t) + \varepsilon^2 S_2(x, t) + \ldots, \tag{2.29}$$

$$\kappa = S_x, \quad \omega = -S_t.$$

Substituting these expansions into eqs. (2.25) and (2.26), equating terms $O(1)$, $O(\varepsilon)$, $O(\varepsilon^2)$, etc. to zero subsequently, we obtain the following sets of recursive equations for the perturbations $A_i$ and $S_i$ ($i = 0, 1, 2, \ldots$):

$$S_{0t}^2 - S_{0x}^2 = 1, \quad \text{(eiconal equation)} \tag{2.30}$$

$$2S_{0t}S_{1t} - 2S_{0x}S_{1x} = 0, \tag{2.31}$$

$$2S_{0t}S_{2t} - 2S_{0x}S_{2x} = \frac{1}{A_0}(A_{0tt} - A_{0xx}) - \tfrac{5}{6}b^2 A_0^2 + S_{1x}^2 - S_{1t}^2, \ \ldots \ etc. \tag{2.32}$$

$$2S_{0x}A_{0x} - 2S_{0t}A_{0t} + (S_{0xx} - S_{0tt})A_0 = 0, \tag{2.33}$$

$$2S_{0x}A_{1x} - 2S_{0t}A_{1t} + (S_{0xx} - S_{0tt})A_1 = 2S_{1t}A_{0t} - 2S_{1x}A_{0x} - (S_{1xx} - S_{1tt})A_0, \ \ldots \ etc. \tag{2.34}$$

For a wavetrain given at $t=0$ we may prescribe the initial conditions:

$$\left.\begin{aligned} S_0(x, 0) &= \sigma(x), \quad S_i(x, 0) \equiv 0 \quad (i = 1, 2, \ldots), \\ A_0(x, 0) &= \alpha(x), \quad A_i(x, 0) \equiv 0 \quad (i = 1, 2, \ldots). \end{aligned}\right\} \tag{2.35}$$

First $S_0(x, t)$ may be solved from (2.30) by using the method of characteristics, then the linear equation (2.33) may be solved for $A_0$ by transforming on characteristic coordinates, which are the same as for the eiconal equation (2.30). Then the homogeneous linear equation (2.31) for

$S_1$ with zero initial condition is seen to have the zero solution: $S_1 \equiv 0$ and from this result follows that also eq. (2.34) for $A_1$ gives $A_1 \equiv 0$. The calculation may proceed further by solving $S_2$ from equation (2.32) and subsequently $A_2, S_3, A_3, \ldots$, from higher order equations. It may be shown easily that in the case of a linearized problem ($b=0$) the results are equivalent to those obtained by the ray method ([1]).

A special property of the equations (2.25), (2.26) and (2.27) is the existence of solutions representing wavetrains with permanent envelopes. To make this clear, $\omega$ is eliminated from these equations to give approximately

$$\kappa_t + C_g \kappa_x + \tfrac{1}{2}\varepsilon^2 \frac{\partial}{\partial x}\left[\frac{1}{\omega_0 A}(A_{tt} - A_{xx}) - \frac{5}{6}\frac{b^2 A^2}{\omega_0}\right] = 0\,, \tag{2.36}$$

$$(\omega_0 A^2)_t + (C_g \omega_0 A^2)_x = 0\,, \quad C_g(\kappa) = \omega_0'(\kappa) = \kappa/\omega_0(\kappa)\,. \tag{2.37}$$

For wavetrains with permanent envelopes we consider solutions of eqs. (2.36) and (2.37) of the form

$$\kappa = \kappa(\xi)\,, \quad A = A(\xi)\,, \quad \xi = x - Ct\,, \tag{2.38}$$

where $C$ will be determined later on. Insertion of these expressions into (2.36) and (2.37) leads to a set of two ordinary differential equations for $\kappa(\xi)$ and $A(\xi)$, of which the second one has been integrated once:

$$(C_g - C)\kappa_\xi + \tfrac{1}{2}\varepsilon^2 \frac{d}{d\xi}\left[\frac{A_{\xi\xi}}{\omega_0 A}(C^2 - 1) - \frac{5}{6}\frac{b^2 A^2}{\omega_0}\right] = 0\,, \tag{2.39}$$

$$(C - C_g)\omega_0 A^2 = \text{constant}\,. \tag{2.40}$$

In order that we have other approximate solutions than the trivial one $A = \text{constant}$, $\kappa = \text{constant}$ (which would be the result from the Whitham theory), the two terms in equation (2.39) should be of the same order of magnitude. To this end we consider wavetrains for which the wavenumber can be written as follows

$$\kappa(\xi) = \bar{\kappa} + \varepsilon\kappa^{(1)}(\xi) + \ldots\,, \quad \bar{\kappa} = \text{constant}\,.$$

Then we have $\kappa'(\xi) = O(\varepsilon)$ and if we choose $C$ to be equal to the linear group velocity corresponding to $\bar{\kappa}$, we also have

$$C_g(\kappa) = C + \varepsilon\omega_0''(\bar{\kappa})\kappa^{(1)} + \ldots\,, \quad C \doteq \omega_0'(\bar{\kappa}) = \bar{\kappa}/\omega_0(\bar{\kappa}) < 1\,.$$

Substituting these relations in equations (2.39) and (2.40) and linearizing with respect to $\varepsilon$, we obtain approximately

$$\omega_0(\bar{\kappa})\omega_0''(\bar{\kappa})\frac{d}{d\xi}\left[(\kappa^{(1)})^2\right] + \frac{d}{d\xi}\left[\frac{A_{\xi\xi}}{A}(C^2 - 1) - \tfrac{5}{6}b^2 A^2\right] = 0\,, \tag{2.41}$$

$$\kappa^{(1)} A^2 = \text{constant}\,. \tag{2.42}$$

Eliminating $\kappa^{(1)}$ from eqs. (2.41), (2.42), performing two integrations and putting $\tilde{E} = A^2$, we get finally

$$(1 - C^2)\tilde{E}_\xi^2 = -\tfrac{5}{3}b^2\tilde{E}^3 + D_2\tilde{E}^2 + D_1\tilde{E} + D_0\,, \tag{2.43}$$

with $D_0$, $D_1$ and $D_2$ arbitrary constants of integration. This is a similar equation as derived by Chu and Mei [8] for water waves on water of infinite depth. Equation (2.43) has periodical solutions in the form of "cnoidal" functions and solitary wave solutions are also possible. Remark that $\tilde{E} = O(1)$ so that the possible modulations of the amplitude are $O(1)$. The variations of the wavenumber, described by $\varepsilon\kappa^{(1)}$, are $O(\varepsilon)$ about the constant value $\bar{\kappa}$. We recall that the Whitham equations in this case only have trivial permanent envelope solutions in the form of uniform wavetrains. An investigation by Ablowitz and Benney [14] has shown that by considering higher order terms in the Whitham theory the modulation equations are modified by terms allowing them to have non-trivial permanent envelope wave solutions as well. However, the possible modulations of $\kappa$ and $A$ should be expected to be both very small in that case.

### 3. Application to Nonlinear Water Waves (Stokes Waves)

In this section we will apply the method of section 2 to the problem of two-dimensional irrotational free surface waves on water of finite depth. Let the undisturbed free surface of the water coincide with the plane $z=0$ of a Cartesian $x$, $y$, $z$-coordinate system, and let the bottom be given by the equation $z=-h=$constant. We will consider two-dimensional wave motions in the fluid, which means that all quantities are independent of one of the horizontal coordinates, $y$ say.

As the basic small parameter of the problem we will use the wave steepness $\varepsilon=\alpha/\lambda\ll 1$, where $\alpha$ and $\lambda$ denote a characteristic amplitude and wavelength, respectively. Since we will consider $\lambda$ to be of order unity, $\varepsilon$ may also serve as a measure for the (small) amplitude of the waves.

Let the equation of the free surface be given by the equation $z=\tilde{\eta}(x, t; \varepsilon)$, then there exists a velocity potential $\tilde{\Phi}(x, z, t; \varepsilon)$ in the region $\tilde{\eta}(x, t)\geqq z\geqq -h$, satisfying the two-dimensional Laplace equation with two free surface conditions at $z=\tilde{\eta}$ and a bottom condition at $z=-h$. As we consider waves with small but finite amplitude, it is useful to introduce the scaled velocity potential $\Phi(x, z, t; \varepsilon)$ and the scaled free surface elevation $\eta(x, t; \varepsilon)$ as follows:

$$\tilde{\eta}(x, t; \varepsilon)=\varepsilon\eta(x, t; \varepsilon)\,, \quad \tilde{\Phi}(x, z, t; \varepsilon)=\varepsilon\Phi(x, z, t; \varepsilon)\,,$$

where $\eta$ and $\Phi$ are assumed to be $O(1)$ as $\varepsilon\to 0$, together with all their partial derivatives up to any order. The non-linear boundary value problem involving $\Phi$ and $\eta$ then becomes ([12], [13]):

$$\Phi_{xx}+\Phi_{zz}=0\,, \quad \varepsilon\eta(x, t; \varepsilon)>z>-h \tag{3.1}$$

$$\left.\begin{aligned}&\Phi_t+g\eta+\tfrac{1}{2}\varepsilon(\Phi_x^2+\Phi_z^2)=0 \quad (3.2)\\ &\eta_t+\varepsilon\eta_x\Phi_x-\Phi_z=0 \quad (3.3)\end{aligned}\right\} \text{ at } z=\varepsilon\eta(x, t; \varepsilon)$$

$$\Phi_z=0\,, \qquad \text{at } z=-h \tag{3.4}$$

where $g$ denotes the acceleration of gravity. For $\varepsilon=0$ this boundary value problem reduces to the well-known linear problem for water waves of infinitesimal amplitude.

Before applying the perturbation method of section 2 it appears to be convenient to rewrite free surface conditions (3.2) and (3.3) as a single condition for $\Phi$ to be satisfied at the undisturbed free surface $z=0$. To this end first $\eta$ is eliminated from condition (3.2) and (3.3) to give a single condition for $\Phi$ to be satisfied at $z=\varepsilon\eta$:

$$\Phi_{tt}+g\Phi_z+2\varepsilon[\Phi_x\Phi_{xt}+\Phi_z\Phi_{zt}]+\varepsilon^2[\Phi_x^2\Phi_{xx}+2\Phi_x\Phi_z\Phi_{xz}+\Phi_z^2\Phi_{zz}]=0\,. \tag{3.5}$$

Using condition (3.2), $\eta$ may be expanded in powers of $\varepsilon$ by means of a Taylor expansion about $z=0$:

$$\eta=-\frac{1}{g}[\Phi_t+\tfrac{1}{2}\varepsilon(\Phi_x^2+\Phi_z^2)]_{z=0}+\frac{\varepsilon}{g^2}[\Phi_t\Phi_{zt}]_{z=0}+O(\varepsilon^2)\,. \tag{3.6}$$

With the aid of this expansion for $\eta$ we may expand condition (3.5) to give the final condition for $\Phi$ to be satisfied at $z=0$:

$$\begin{aligned}&\Phi_{tt}+g\Phi_z+\varepsilon\left[2\Phi_x\Phi_{xt}+2\Phi_z\Phi_{zt}-\frac{1}{g}\Phi_t\Phi_{ttz}-\Phi_t\Phi_{zz}\right]+\\ &+\varepsilon^2\left[\Phi_x^2\Phi_{xx}+2\Phi_x\Phi_z\Phi_{xz}+\Phi_z^2\Phi_{zz}+(\Phi_{ttz}+g\Phi_{zz})\left(\frac{1}{g^2}\Phi_t\Phi_{zt}-\frac{1}{2g}\Phi_x^2-\frac{1}{2g}\Phi_z^2\right)+\right.\\ &\left.-\frac{2}{g}\Phi_t(\Phi_{xz}\Phi_{xt}+\Phi_x\Phi_{xzt}+\Phi_{zz}\Phi_{zt}+\Phi_z\Phi_{zzt})\right]+O(\varepsilon^3)=0\,, \text{ at } z=0\,.\end{aligned} \tag{3.7}$$

In order to study slowly varying wavetrain solutions with modulation rate $O(\varepsilon)$ per wavelength or period, we introduce the slow variables $x^*$ and $t^*$:

$$x^* = \varepsilon x\,, \quad t^* = \varepsilon t$$

and look for solutions $\Phi$ of the form

$$\Phi = \phi[\varepsilon^{-1}S(x^*, t^*; \varepsilon), x^*, z, t^*; \varepsilon]\,, \tag{3.8}$$

where $\phi$ is a periodic function of the variable $p=\varepsilon^{-1}S(x^*, t^*; \varepsilon)$ with period normalized to $2\pi$. Omitting the asterisks, we then find for $\phi$ the equation

$$\kappa^2\phi_{pp}+\phi_{zz} = -2\varepsilon\kappa\phi_{px}-\varepsilon\kappa_x\phi_p-\varepsilon^2\phi_{xx}\,, \quad 0>z>-h \tag{3.9}$$

with $\kappa=S_x$ and $\omega=-S_t$ as notations for the local wavenumber and local frequency, respectively. The bottom condition for $\phi$ becomes:

$$\phi_z = 0\,, \text{ at } z=-h \tag{3.10}$$

and the free surface condition (3.7) for $\Phi$ transforms into the following condition for $\phi$ at $z=0$:

$$\begin{aligned}\omega^2\phi_{pp}+g\phi_z = \varepsilon\Big[&2\omega\phi_{pt}+\omega_t\phi_p+2\omega\kappa^2\phi_p\phi_{pp}+2\omega\phi_z\phi_{pz}-\frac{\omega^3}{g}\phi_p\phi_{ppz}-\omega\phi_p\phi_{zz}\Big]+\\ +\varepsilon^2\Big[&-\phi_{tt}-2\kappa\kappa_t\phi_p^2-2\kappa^2\phi_p\phi_{pt}+2\omega\kappa\phi_p\phi_{px}+\\ &+2\omega\kappa\phi_x\phi_{pp}-2\phi_z\phi_{zt}+2\frac{\omega^2}{g}\phi_p\phi_{ptz}+\frac{\omega\omega_t}{g}\phi_p\phi_{pz}+\\ &+\frac{\omega^2}{g}\phi_t\phi_{ppz}+\phi_t\phi_{zz}-\kappa^4\phi_p^2\phi_{pp}-2\kappa^2\phi_p\phi_z\phi_{pz}-\phi_z^2\phi_{zz}+\\ &-(\omega^2\phi_{ppz}+g\phi_{zz})\left(\frac{\omega^2}{g^2}\phi_p\phi_{pz}-\frac{\kappa^2}{2g}\phi_p^2-\frac{1}{2g}\phi_z^2\right)+\\ &-\frac{2\omega^2\kappa^2}{g}\phi_p(\phi_z\phi_{ppp}-\phi_p\phi_{ppz})\Big]+O(\varepsilon^3)=0\,.\end{aligned} \tag{3.11}$$

For $\varepsilon=0$ a solution for $\phi$ being periodic in $p$ is easily found by means of separation of variables:

$$\phi = a\cosh\left[(z+h)\frac{g\kappa^2\operatorname{tgh}\kappa h}{\omega^2}\right]\sin\left[p\frac{g\kappa\operatorname{tgh}\kappa h}{\omega^2}\right]. \tag{3.12}$$

The normalization of the period to $2\pi$ yields the dispersion relation for linear water waves:

$$\omega^2 = \omega_0^2(\kappa) = g\kappa\operatorname{tgh}\kappa h\,. \tag{3.13}$$

For small positive values of $\varepsilon$ we still expect the problem to possess solutions periodic in $p$, however, the period, which by (3.12) is found to be

$$\frac{2\pi\omega^2}{g\kappa\operatorname{tgh}\kappa h}$$

in the reduced case $\varepsilon=0$, will in general be different in the non-reduced case $\varepsilon>0$. Therefore the non-reduced period is assumed to be of the form:

$$\frac{2\pi\omega^2}{g\kappa\operatorname{tgh}\kappa h+\varepsilon\Omega_1+\varepsilon^2\Omega_2+\dots}$$

where the $\Omega_i$ $(i=1, 2, \dots)$ denote unknown functions of $x$, $t$ and $\varepsilon$, being $O(1)$ as $\varepsilon\to 0$. The normalization of the non-reduced period to $2\pi$ gives the following expansion for $\omega^2$:

$$\omega^2 \sim \omega_0^2(\kappa)+\varepsilon\Omega_1+\varepsilon^2\Omega_2+\dots\,, \quad \omega_0^2(\kappa)=g\kappa\operatorname{tgh}\kappa h\,. \tag{3.14}$$

This is the appropriate generalization of the linear dispersion relation (3.13) to non-linear waves with modulation rate $O(\varepsilon)$ per wavelength or period. The functions $\Omega_1, \Omega_2, \ldots$ will be needed later on to suppress secular terms.

Analogous to the method of the preceding section, $\phi$ is expanded as follows:

$$\phi \sim \phi_0+\varepsilon\phi_1+\varepsilon^2\phi_2+\ldots, \tag{3.15}$$

with $\phi_0, \phi_1, \phi_2, \ldots$ functions of $p, x, z, t$ and $\varepsilon$, being $O(1)$ as $\varepsilon\to 0$ together with all their partial derivatives up to any order. Substituting the expansion (3.15) into eq. (3.9) for $\phi$ and equating to zero the contributions of $O(1)$, $O(\varepsilon)$ and $O(\varepsilon^2)$, respectively, we get the following equations for $\phi_0$, $\phi_1$ and $\phi_2$:

$$\kappa^2\phi_{0pp}+\phi_{0zz}=0, \tag{3.16}$$

$$\kappa^2\phi_{1pp}+\phi_{1zz}=-2\kappa\phi_{0px}-\kappa_x\phi_{0p}, \tag{3.17}$$

$$\kappa^2\phi_{2pp}+\phi_{2zz}=-2\kappa\phi_{1px}-\kappa_x\phi_{1p}-\phi_{0xx}. \tag{3.18}$$

The bottom condition (3.10) leads to

$$\phi_{iz}=0, \quad i=0, 1, 2, \ldots, \text{at } z=-h, \tag{3.19}$$

and the free surface condition (3.11) gives the following conditions for $\phi_0$, $\phi_1$ and $\phi_2$ to be satisfied at $z=0$:

$$\omega_0^2\phi_{0pp}+g\phi_{0z}=0, \tag{3.20}$$

$$\begin{aligned}\omega_0^2\phi_{1pp}+g\phi_{1z}= & -\Omega_1\phi_{0pp}+2\omega\phi_{0pt}+\omega_t\phi_{0p}+2\omega\kappa^2\phi_{0p}\phi_{0pp}+\\ & +2\omega\phi_{0z}\phi_{0pz}-\frac{\omega^3}{g}\phi_{0p}\phi_{0ppz}-\omega\phi_{0p}\phi_{0zz},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\omega_0^2\phi_{2pp}+g\phi_{2z}= & -\Omega_1\phi_{1pp}-\Omega_2\phi_{0pp}+2\omega\phi_{1pt}+\omega_t\phi_{1p}+\\ & +2\omega\kappa^2(\phi_{0p}\phi_{1pp}+\phi_{1p}\phi_{0pp})+2\omega(\phi_{0z}\phi_{1pz}+\phi_{1z}\phi_{0pz})+\\ & -\frac{\omega^3}{g}(\phi_{0p}\phi_{1ppz}+\phi_{1p}\phi_{0ppz})-\omega(\phi_{0p}\phi_{1zz}+\phi_{1p}\phi_{0zz})+\\ & -\phi_{0tt}-2\kappa\kappa_t\phi_{0p}^2-2\kappa^2\phi_{0p}\phi_{0pt}+2\omega\kappa(\phi_{0p}\phi_{0px}+\phi_{0x}\phi_{0pp})+\\ & -2\phi_{0z}\phi_{0zt}+2\frac{\omega^2}{g}\phi_{0p}\phi_{0ptz}+\frac{\omega\omega_t}{g}\phi_{0p}\phi_{0pz}+\frac{\omega^2}{g}\phi_{0t}\phi_{0ppz}+\\ & +\phi_{0t}\phi_{0zz}-\kappa^4\phi_{0p}^2\phi_{0pp}-2\kappa^2\phi_{0p}\phi_{0z}\phi_{0pz}-\phi_{0z}^2\phi_{0zz}+\\ & +(\omega^2\phi_{0ppz}+g\phi_{0zz})\left(\frac{1}{2g}\phi_{0z}^2+\frac{\kappa^2}{2g}\phi_{0p}^2-\frac{\omega^2}{g^2}\phi_{0p}\phi_{0pz}\right)+\\ & +\frac{2\omega^2\kappa^2}{g}\phi_{0p}(\phi_{0p}\phi_{0ppz}-\phi_{0z}\phi_{0ppp}),\end{aligned} \tag{3.22}$$

where the expansion (3.14) for $\omega^2$ has been inserted only in the lefthand side of condition (3.11) in order to avoid unnecessary computational work.

The general solution for $\phi_0$ being periodic in $p$ with period $2\pi$ is given by

$$\phi_0=a(x, t; \varepsilon)\cosh[\kappa(h+z)]\sin p+\psi(x, t; \varepsilon), \tag{3.23}$$

where the unknown functions $a$ and $\psi$ are $O(1)$ as $\varepsilon\to 0$, together with all their partial derivatives up to any order. Without loss of generality we may discard solutions proportional to $\cosh[\kappa(h+z)]\cos p$, which may be absorbed in $\phi_0$ by a suitable shift of $p$. Furthermore, by allowing the potential amplitude function $a$ to depend on $\varepsilon$ we have the freedom to require that terms proportional to $\cosh[\kappa(h+z)]\sin p$ should not occur in the higher order solutions $\phi_1$, $\phi_2$, etc.

Using expression (3.23) for $\phi_0$, we get for $\phi_1$ the equation

$$\kappa^2\phi_{1pp}+\phi_{1zz} = -(\kappa_x a+2\kappa a_x)\cosh[\kappa(h+z)]\cos p - 2\kappa\kappa_x a(h+z)\sinh[\kappa(h+z)]\cos p\,. \tag{3.24}$$

A particular solution of (3.24), satisfying bottom condition (3.19) and being periodic in $p$ with period $2\pi$ is given by

$$\phi_1^P = -\tfrac{1}{2}a\kappa_x(z+h)^2\cosh[\kappa(h+z)]\cos p - a_x(h+z)\sinh[\kappa(h+z)]\cos p\,. \tag{3.25}$$

Putting $\phi_1 = \tilde{\phi}_1+\phi_1^P$, we get for $\tilde{\phi}_1$ the homogeneous equation corresponding to equation (3.24):

$$\kappa^2\tilde{\phi}_{1pp}+\tilde{\phi}_{1zz} = 0\,, \tag{3.26}$$

with the bottom condition

$$\tilde{\phi}_{1z} = 0 \text{ at } z=-h \tag{3.27}$$

and the free surface condition at $z=0$:

$$\begin{aligned}\omega_0^2\tilde{\phi}_{1pp}+g\tilde{\phi}_{1z} = {} & [a\omega_t\cosh\kappa h+2\omega(a\cosh\kappa h)_t-\omega_0^2 a_x h\sinh\kappa h+ \\ & +ga\kappa_x h\cosh\kappa h+ga_x(\sinh\kappa h+\kappa h\cosh\kappa h)]\cos p+ \\ & +a\Omega_1\cosh\kappa h\sin p-\omega\kappa^2a^2\sin 2p+ \\ & +\tfrac{1}{2}\kappa a^2\omega\left(\frac{\omega^2}{g}\sinh\kappa h-\cosh\kappa h\right)\cosh\kappa h\sin 2p\,.\end{aligned} \tag{3.28}$$

It may be verified that the term with $\cos p$ in the righthand side of (3.28) gives rise to a solution proportional to

$$p\sin p\cosh[\kappa(h+z)]-\kappa(z+h)\cos p\sinh[\kappa(h+z)]\,.$$

This kind of solution is not periodic in $p$ and cannot occur in our solution. Hence the coefficient of $\cos p$ in condition (3.28) should be expanded analogous to the method of section 2:

$$\begin{aligned}& a\omega_t\cosh\kappa h+2\omega(a\cosh\kappa h)_t-\omega_0^2a_xh\sinh\kappa h+ga\kappa_xh\cosh\kappa h+ \\ & \qquad +ga_x(\sinh\kappa h+\kappa h\cosh\kappa h)\sim\varepsilon\Lambda_1+\varepsilon^2\Lambda_2+\ldots = O(\varepsilon^2)\end{aligned} \tag{3.29}$$

in order to remove the secular $\cos p$ term from (3.28) and at the same time to be able to suppress similar terms in higher order problems by chosing the unknown functions $\Lambda_1$, $\Lambda_2$, ..., in a suitable way. Note that we have taken $\Lambda_1\equiv 0$, since no secular $\cos p$ terms will arise in the boundary value problem for $\phi_2$. In a similar way one shows that the term with $\sin p$ in condition (3.28) is a secular term and hence it must vanish. This gives:

$$\Omega_1 = 0\,, \tag{3.30}$$

which means that

$$\omega^2 = \omega_0^2(\kappa)+O(\varepsilon^2)\,. \tag{3.31}$$

In order to give a physical interpretation of eq. (3.29) the "physical" amplitude $A(x, t; \varepsilon)$ is introduced as the coefficient of $\cos p$ in the expansion of the free surface elevation $\eta(p, x, t; \varepsilon)$. According to eq. (3.6) and using (3.31) the expansion of $\eta$ in terms of $\phi$ appears to be

$$\eta = \left[\frac{\omega_0}{g}\phi_{0p}\right]_{z=0}-\frac{\varepsilon}{g}\left[\phi_{0t}+\tfrac{1}{2}\kappa^2\phi_{0p}^2+\tfrac{1}{2}\phi_{0z}^2-\frac{\omega_0^2}{g}\phi_{0p}\phi_{0pz}-\omega_0\phi_{1p}\right]_{z=0}+O(\varepsilon^2)\,. \tag{3.32}$$

Using expression (3.23) for $\phi_0$ it is seen that the physical amplitude $A$ is related to the potential amplitude $a$ as follows

$$A = \frac{\kappa a}{\omega_0} \sinh \kappa h\,. \tag{3.33}$$

Equation (3.29) may be expressed in terms of $A$, and upon use of (3.33) and (3.31) it may be shown after some calculations to be equivalent to

$$E_t + [\omega_0'(\kappa)E]_x + O(\varepsilon^2) = 0\,, \quad E = \frac{A^2}{\omega_0}\,. \tag{3.34}$$

This equation has the physical interpretation of a conservation equation for the "energy density" $E$ propagating with linear group velocity $\omega_0'(\kappa)$.

The boundary value problem for $\tilde{\phi}_1$ may now be solved to give

$$\tilde{\phi}_1 = \frac{3a^2\kappa^2}{8\omega_0 \sinh^2 \kappa h} \cosh[2\kappa(h+z)] \sin 2p + \psi^*(x, t; \varepsilon) + O(\varepsilon^2)\,, \tag{3.35}$$

where the additional function $\psi^*$ is still undetermined; it appears to be determined by the suppression of a secular term linear in $p$ in the boundary value problem for $\phi_3$ and hence it is of no further concern for us as we will break off after the determination of the secular terms in the problem for $\phi_2$, which problem does not involve $\psi^*$. The unspecified terms $O(\varepsilon^2)$ in $\tilde{\phi}_1$ are a result of the replacement of $\omega^2$ by $\omega_0^2 + O(\varepsilon^2)$ during the calculation and they do not play a role in the problem for $\phi_2$.

We now may write down the equation for $\phi_2$, using the expressions found for $\phi_0$ and $\phi_1$. We get:

$$\begin{aligned} \kappa^2 \phi_{2pp} + \phi_{2zz} = {} & -\psi_{xx} + [\ldots] \cos 2p - a_{xx} \cosh[\kappa(h+z)] \sin p + \\ & -(3a_x\kappa_x + a\kappa_{xx} + 2\kappa a_{xx})(h+z) \sinh[\kappa(h+z)] \sin p + \\ & -(\tfrac{3}{2}a\kappa_x^2 + a\kappa\kappa_{xx} + 3\kappa a_x\kappa_x)(h+z)^2 \cosh[\kappa(h+z)] \sin p + \\ & -a\kappa\kappa_x^2(z+h)^3 \sinh[\kappa(h+z)] \sin p + O(\varepsilon^2)\,, \end{aligned} \tag{3.36}$$

where the terms with cos $2p$ have been left unspecified since they do not give rise to secular terms in $\phi_2$. A particular solution $\phi_2^P$ of eq. (3.36), satisfying bottom condition (3.19) is given by

$$\begin{aligned} \phi_2^P = {} & \{-\tfrac{1}{2}a_{xx}(h+z)^2 \cosh[\kappa(h+z)] - (\tfrac{1}{6}a\kappa_{xx} + \tfrac{1}{2}a_x\kappa_x)(h+z)^3 \sinh[\kappa(h+z)] + \\ & -\tfrac{1}{8}a\kappa_x^2(h+z)^4 \cosh[\kappa(h+z)]\} \sin p - \tfrac{1}{2}\psi_{xx}(z+h)^2 + [\ldots] \cos 2p + O(\varepsilon^2)\,. \end{aligned} \tag{3.37}$$

Putting $\phi_2 = \tilde{\phi}_2 + \phi_2^P$ we get for $\tilde{\phi}_2$ the same homogeneous equation as for $\tilde{\phi}_1$, with the same bottom condition (3.27). The free surface condition for $\tilde{\phi}_2$ becomes at $z=0$:

$$\begin{aligned} \omega_0^2 \tilde{\phi}_{2pp} + g\tilde{\phi}_{2z} = {} & a \cosh \kappa h[\Omega_2 + F_1 + F_2] \sin p + [gh\psi_{xx} - \psi_{tt} + F_3] + \\ & + [\ldots] \cos 2p + [\ldots] \cos 3p + O(\varepsilon^2)\,, \end{aligned} \tag{3.38}$$

with the abbreviations:

$$\begin{aligned} F_1 = {} & -2\omega_0\kappa\psi_x + \frac{\kappa^2\psi_t}{\cosh^2 \kappa h} - a^2\kappa^4 \cosh 2\kappa h \left(\frac{9}{8 \sinh^2 \kappa h} + \frac{1}{4 \cosh^2 \kappa h}\right) + \\ & + a^2\kappa^4(\tfrac{11}{4} - \cosh^2 \kappa h) + O(\varepsilon^2)\,, \end{aligned} \tag{3.39}$$

$$\begin{aligned} aF_2 = {} & -\frac{(a \cosh \kappa h)_{tt}}{\cosh \kappa h} + \omega_0 h^2(a\kappa_x)_t + 2\omega_0 a_x\kappa_t h^2 + \tfrac{1}{2}\kappa_x\omega_{0t}ah^2 + \\ & + [\omega_0 a\kappa_x\kappa_t h^3 + 2\omega_0 a_{xt}h + a_x\omega_{0t}h] \,\mathrm{tgh}\, \kappa h + \\ & + g(\tfrac{1}{6}a\kappa_{xx} + \tfrac{1}{2}a_x\kappa_x)\left(\frac{\kappa h^3}{\cosh^2 \kappa h} + 3h^2 \,\mathrm{tgh}\, \kappa h\right) + gha_{xx} + \tfrac{1}{2}ga\kappa_x^2h^3 + O(\varepsilon^2)\,, \end{aligned} \tag{3.40}$$

$$F_3 = -\tfrac{1}{2}\omega_0[aa_x\kappa^2 h \operatorname{tgh}\kappa h - a^2\kappa_x\cosh^2\kappa h - aa_x\kappa(1+\cosh^2\kappa h) + $$
$$- a^2\kappa\kappa_x h\cosh\kappa h\sinh\kappa h] - \tfrac{1}{2}(a\cosh\kappa h)_t a\kappa^2\left(\cosh\kappa h + \frac{\sinh^2\kappa h}{\cosh\kappa h}\right) +$$
$$- \kappa\kappa_t a^2\cosh^2\kappa h + \frac{\omega_0\omega_{0t}}{2g}a^2\kappa\cosh\kappa h\sinh\kappa h + O(\varepsilon^2)\,. \tag{3.41}$$

The term $F_1$ does not contain derivatives of $\kappa$ and $a$, whereas $F_2$, containing derivatives of $\kappa$ and $a$, is the contribution of the particular solution $\phi_2^P$.

The righthand side of condition (3.38) contains two secular terms: the term with $\sin p$, yielding the same type of nonperiodic solutions as the secular terms in the free surface condition (3.28) for $\tilde{\phi}_1$, and the term involving $F_3$, which gives rise to secular terms in the following way. Putting

$$\tilde{\phi}_2 = \operatorname{Re}[f(\chi)]\,,\quad \chi = \frac{p}{\kappa} + \mathrm{i}z\,,\quad f = \tilde{\phi}_2 + \mathrm{i}\zeta\,,$$

then $f(\chi)$ is an analytic function of the complex variable $\chi$ which has to be bounded in the strip $-h \leqq \operatorname{Im}\chi \leqq 0$. If we only consider a constant term, $\sigma$ say, in the righthand side of the free surface condition (3.38), then this condition may be written as:

$$\omega_0^2\tilde{\phi}_{2pp} + g\tilde{\phi}_{2z} = \omega_0^2\tilde{\phi}_{2pp} - g\kappa\zeta_p = \sigma\,,\quad \text{at } z=0\,.$$

Integration with respect to $p$ then gives:

$$\omega_0^2\tilde{\phi}_{2p} - g\kappa\zeta = \sigma p + \text{constant}\,,\quad \text{at } z=0\,.$$

Now both $\tilde{\phi}_{2p}$ and $\zeta$ should be bounded for all $p$ at $z=0$, so $\sigma p$ is a secular term.

Equating both secular terms in condition (3.38) to zero, we get:

$$\Omega_2 + F_1 + F_2 = 0\,, \tag{3.42}$$
$$gh\psi_{xx} - \psi_{tt} + F_3 = 0\,. \tag{3.43}$$

Hence we find from (3.42) and (3.14) the dispersion equation:

$$\omega = \omega_0(\kappa) + \frac{\varepsilon^2\Omega_2}{2\omega_0} + \ldots = \omega_0 - \frac{\varepsilon^2}{2\omega_0}(F_1+F_2) + O(\varepsilon^3)\,.$$

By some straightforward calculations this relation may be shown to be equivalent to

$$\omega = \omega_0(\kappa) + \varepsilon^2\kappa\beta + \frac{\varepsilon^2 gb\kappa^2}{2\omega_0\cosh^2\kappa h} + \tfrac{1}{2}g\kappa^3\frac{\varepsilon^2A^2}{\omega_0}D_0 - \frac{\varepsilon^2}{2\omega_0}F_2 + O(\varepsilon^3) \tag{3.44}$$

with

$$D_0 = \frac{9\operatorname{tgh}^4\kappa h - 10\operatorname{tgh}^2\kappa h + 9}{8\operatorname{tgh}^3\kappa h}$$

and where $\beta = \psi_x$ denotes the mean velocity and $\varepsilon b$, $b = O(1)$, denotes the mean wave height which by (3.32) is related to $\psi_t$ as follows:

$$gb = -\psi_t - \frac{g\kappa A^2}{2\sinh 2\kappa h} + O(\varepsilon^2)\,. \tag{3.45}$$

Another straightforward calculation shows that equation (3.43) may be written as

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x}(\beta h + \tfrac{1}{2}g\kappa E) + O(\varepsilon^2) = 0\,, \tag{3.46}$$

which may be interpreted physically as an averaged equation expressing the conservation of mass.

By elimination of $\omega$ and $\psi$ we may write down the final results as follows:

$$\frac{\partial \kappa}{\partial t} + \frac{\partial}{\partial x}\left[\omega_0 + \varepsilon^2 \beta\kappa + \frac{\varepsilon^2 g b \kappa^2}{2\omega_0 \cosh^2 \kappa h} + \tfrac{1}{2} g\kappa^3 \frac{\varepsilon^2 A^2}{\omega_0} D_0 - \frac{\varepsilon^2 F_2}{2\omega_0}\right] + O(\varepsilon^3) = 0\,, \qquad (3.47)$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\omega_0'(\kappa)E\right] + O(\varepsilon^2) = 0\,, \quad E = \frac{A^2}{\omega_0}\,, \qquad (3.48)$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x}\left[g b + \frac{g\kappa A^2}{2 \sinh 2\kappa h}\right] + O(\varepsilon^2) = 0\,, \qquad (3.49)$$

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x}\left[\beta h + \tfrac{1}{2} g\kappa E\right] + O(\varepsilon^2) = 0\,. \qquad (3.50)$$

These four equations constitute a set of equations to determine the four slowly varying parameters $\kappa$ (wave number), $A$ (wave amplitude), $b$ (mean wave height) and $\beta$ (mean fluid velocity) as functions of the slow variables $x$ and $t$. The only difference with the results obtained by Whitham's method is the appearance of the extra term involving the rather complicated expression $F_2$ in the dispersion equation (3.47). For very slowly modulated wavetrains the derivatives of $a$ and $\kappa$ appearing in $F_2$ are small compared to $a$ and $\kappa$ themselves. In that case $F_2$ may be neglected and Whitham's results are found again. The implications of the additional terms compared to Whitham's results are discussed fully in the two papers of Chu and Mei [8], [9].

We finally note that for the case of water of infinite depth only two equations involving $\kappa$ and $A$ remain:

$$\frac{\partial \kappa}{\partial t} + \frac{\partial}{\partial x}\left[\omega_0(\kappa) + \tfrac{1}{2}\varepsilon^2 \omega_0 \kappa^2 A^2 + \frac{\varepsilon^2}{2A}\left(\frac{A}{\omega_0}\right)_{tt}\right] + O(\varepsilon^3) = 0\,,$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\omega_0'(\kappa)E\right] + O(\varepsilon^2) = 0\,, \quad E = \frac{A^2}{\omega_0}\,, \quad \omega_0^2 = g\kappa\,.$$

These equations are discussed also in [8], [9], where it is shown among others that they allow periodic solutions representing periodically modulated wavetrains which are not possible within the scope of the Whitham theory.

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